

Central tetrads and quantum spacetimes

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In this paper we perform a parallel analysis to the model proposed in [25]. By considering the central co-tetrad (instead of the central metric) we investigate the modifications in the gravitational metrics coming from the noncommutative spacetime of the κ -Minkowski type in four dimensions. The differential calculus corresponding to a class of Jordanian κ -deformations provides metrics which lead either to cosmological constant or spatial-curvature type solutions of non-vacuum Einstein equations. Among vacuum solutions we find pp-wave type.

I. INTRODUCTION

The quantum gravity effects at the Planck scale might modify the structure of spacetime leading to its noncommutativity [1, 2]. From algebraic point of view in the quantum phase space, besides the non trivial Heisenberg relations between coordinates and momenta, the coordinate relations will be modified and one has to introduce noncommutativity of coordinates themselves [3, 4]. Such modification of spacetime might have influence on physical solutions, e.g. in gravitational and cosmological effects [5–8]. The (noncommutative) modification of spacetime should be therefore included in the theoretical predictions for (astrophysical) measurements [9]. Understandably any corrections to classical solutions would be of the order of the Planck scale which makes them difficult to detect with today's technology. However the theoretical models can suggest new directions to be developed. The finding of any falsifiable prediction to be tested in the real (astrophysical) experiments and observations would be very important in the experimental search for quantum gravity effects and high energy physics. The considerations on deformation of gravitational solutions as well as on cosmological effects coming from noncommutativity are also very timely due to LIGO and PLANCK experiments.

In noncommutative spacetimes approach it is assumed that effects of noncommutativity should be visible in quantum gravity and would allow us to model these in an effective description without full knowledge of quantum gravity itself. One of the most known types of noncommutative spacetime is when coordinates satisfy the Lie algebra type commutation relations. Such deformation was inspired by the κ -deformed Poincaré algebra [10] as deformed symmetry for the κ -Minkowski spacetime [3, 4].

The investigations proposed in this paper focus on κ type of noncommutativity, where the κ -Minkowski commutation relations are as follows:

$$[\hat{x}^i, \hat{x}^j] = 0, \quad [\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa} \hat{x}^i, \quad (1)$$

where κ is the deformation parameter usually related to some quantum gravity scale or Planck mass [3, 10], for example certain bounds on this scale were found in [11, 12].

To study the modifications in the gravitational effects coming from this deformed spacetime one needs to introduce the appropriate differential calculus (compatible with such noncommutativity). There are few approaches in constructing the deformed differential calculi. For example there are the twisted approach [13] and the bicovariant differential calculi formulation based on quantum groups framework [14]. Moreover it has been shown that in the case of time-like κ -Minkowski spacetime the four-dimensional bicovariant differential calculi compatible with κ -Poincaré algebra does not exist, but one can construct a five-dimensional one, which is bicovariant [15–17]. On the other hand considering light-like version of κ -Minkowski spacetime the differential calculus can be bicovariant and four-dimensional [18, 19]. Differential calculi of classical dimension (number of basis one-forms equal to number of coordinates) compatible with κ -Minkowski algebra (for time-, light- and spacelike deformations) were classified in [19]. These differential calculi

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are bicovariant with respect to other (larger) symmetries than κ -Poincaré algebra (except for the light-like case). Alternative approaches to differential calculus on κ -Minkowski space-time were also considered in [20–24].

Our aim in this paper, inspired by the papers [25, 26], is to investigate the noncommutative (quantum) metrics coming from the families of differential calculi introduced in [22] (also included in [19]).

In [25, 26] the authors have investigated the possible noncommutative metrics g , which belong to the center of the κ -Minkowski algebra (1) for certain differential calculi, see also [27]. This condition, necessary in the noncommutative Riemannian geometry to allow contractions, defines set of equations for coefficients in the metric. In the classical limit when $\kappa \rightarrow \infty$ the influence of the noncommutativity remains in the form of the metric and leads to modifications in the known solutions in GR. In this approach authors were able to identify the Einstein tensor built from the central metric with that of the perfect fluid for positive pressure, zero density, and for negative pressure and positive density [25]. In a follow up paper [26] they showed that dark energy (cosmological constant) case can be obtained from the algebraic constraint stemming from the central metric approach.

Our aim in this paper is to consider the central tetrad fields ω^a in the κ -Minkowski algebra (1) instead of the central metric. This way the metric $g = \eta_{ab}\omega^a \otimes \omega^b$ has a Lorentzian signature by definition provided that the flat metric η_{ab} has the same signature. In the central metric formalism one has to impose the Lorentzian signature condition as an additional constraint.

Once we calculate the tetrads related with certain differential calculus (compatible with the κ -Minkowski algebra) we can consider the corresponding gravitational metric in the classical limit as induced from noncommutativity. Classical limit is obtained by $\kappa \rightarrow \infty$ and then noncommutative objects (coordinates, differentials etc.) will become commutative as follows:

$$\begin{aligned} \hat{x}^k &\longrightarrow x^k \quad , \quad \hat{x}^0 \longrightarrow t \\ \hat{\xi}^k &\longrightarrow dx^k \quad , \quad \hat{\xi}^0 \longrightarrow dt \\ g &= \hat{g}_{\mu\nu} \hat{\xi}^\mu \otimes \hat{\xi}^\nu \longrightarrow g_{\mu\nu} dx^\mu \otimes dx^\nu \end{aligned} \quad (2)$$

Of course, we impose that the metric derived from central tetrad in the classical limit has to satisfy Einstein equations.

Then we focus on the non-vacuum Einstein equations in orthonormal tetrad form $G^{ab} = 8\pi G T^{ab}$ with $G^{ab} = R^{ab} - \frac{1}{2}R\eta^{ab}$ and the energy momentum tensor $T^{ab} = (\rho, p, p, p)$ corresponding to the perfect isotropic and barotropic fluid. In cosmology, the equation of state of a perfect fluid is characterized by a dimensionless number, the so-called barotropic factor w equal to the ratio of its pressure p to the energy density ρ : $w = p/\rho$. For example the most known cases are: $w = -1$ (cosmological constant or dark energy), $w = 0$ (dust or dark matter), $w = 1/3$ (radiation) and $w = 1$ (stiff matter). The value $w = -1/3$ corresponds to spatial curvature and separates two cases: for $w \geq -1/3$ the strong energy condition $\rho + 3p \geq 0$ is preserved, for $w < -1/3$ it is violated. The last case characterizes accelerating universe while the former decelerating one.

The main result of this paper is that the effects of the noncommutativity are encoded in the constraints coming from the central tetrad formalism which induces a very special and generic classical solutions: universe with a spatial curvature type of barotropic factor $w = -\frac{1}{3}$ and a universe with dark energy (cosmological constant) with barotropic factor $w = -1$.

II. κ -MINKOWSKI ALGEBRA AND RELATED QUANTUM DIFFERENTIAL CALCULUS

A. κ -Minkowski algebra

κ -Minkowski algebra $\hat{\mathcal{A}} = \mathbb{C}[\hat{x}^\mu]/\mathcal{I}$ with $\mu = 0, \dots, 3$ where \mathcal{I} is a two-sided ideal generated by the commutation relations

$$[\hat{x}^i, \hat{x}^j] = 0, \quad [\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa} \hat{x}^i, \quad (3)$$

where κ is the deformation parameter usually related to some quantum gravity scale or Planck mass. Eq. (3) represents the time-like deformations of the usual Minkowski space. We can also look at more general Lie algebraic deformations of Minkowski space

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}{}_\lambda \hat{x}^\lambda, \quad (4)$$

where $\hat{x}^\mu = (\hat{x}^0, \hat{x}^i)$ and structure constants $C^{\mu\nu}{}_\lambda$ satisfy

$$C^{\mu\alpha}{}_\beta C^{\nu\lambda}{}_\alpha + C^{\nu\alpha}{}_\beta C^{\lambda\mu}{}_\alpha + C^{\lambda\alpha}{}_\beta C^{\mu\lambda}{}_\alpha = 0. \quad (5)$$

$$C^{\mu\nu}{}_{\lambda} = -C^{\nu\mu}{}_{\lambda}. \quad (6)$$

For $C^{\mu\nu}{}_{\lambda} = a^{\mu}\delta_{\lambda}^{\nu} - a^{\nu}\delta_{\lambda}^{\mu}$ we get

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i(a^{\mu}\hat{x}^{\nu} - a^{\nu}\hat{x}^{\mu}). \quad (7)$$

Generally, $a^{\mu} = \frac{1}{\kappa}u^{\mu}$, where a fixed vector $u^{\mu} \in \mathbb{R}^4$ belongs to the classical space (Minkowski space). For $u^{\mu} = (1, \vec{0})$ we get back to eq. (3) as a special case. It turns out to be the most general case either since by linear change of generators one can always transform relations (7) into (3) (cf. [28]). We should be also aware that these relations and hence corresponding noncommutative algebras representing quantum spacetimes are metric independent. Dependence of a metric may come from covariance property under suitable quantum group action. For example, taking the κ -Poincaré (quantum) group with metric of Lorentzian signature $(-, +, +, +)$ one has to distinguish three cases, where $u^2 = -1$ for time-like deformations, $u^2 = 0$ for light-like deformations and $u^2 = 1$ for space-like deformations.

B. Differential calculus of classical dimension

We denote the algebra of differential 1-forms as $d\hat{x}^{\mu} \in \Omega^1(\hat{\mathcal{A}})$ with $d : \hat{\mathcal{A}} \rightarrow \Omega^1(\hat{\mathcal{A}})$. We define the basis 1-forms $d\hat{x}^{\mu} \equiv \hat{\xi}^{\mu}$ in a usual way, where d is the exterior derivative with the property $d^2 = 0$ and satisfies Leibniz rule.

In [19] the construction of the most general algebra of differential one-forms $\hat{\xi}^{\mu}$ compatible with κ -Minkowski algebra (3) that is closed in differential forms (the differential calculus is of classical dimension) is presented. The commutators between one forms and coordinates are given by

$$[\hat{\xi}^{\mu}, \hat{x}^{\nu}] = iK^{\mu\nu}{}_{\alpha}\hat{\xi}^{\alpha}, \quad (8)$$

where $K^{\mu\nu}{}_{\alpha} \in \mathbb{R}$ and after imposing super-Jacobi identities and compatibility condition¹ one gets two constraints on $K^{\mu\nu}{}_{\alpha}$

$$K^{\lambda\mu}{}_{\alpha}K^{\alpha\nu}{}_{\rho} - K^{\lambda\nu}{}_{\alpha}K^{\alpha\mu}{}_{\rho} = C^{\mu\nu}{}_{\beta}K^{\lambda\beta}{}_{\rho}. \quad (9)$$

$$K^{\mu\nu}{}_{\alpha} - K^{\nu\mu}{}_{\alpha} = C^{\mu\nu}{}_{\alpha}. \quad (10)$$

There are only four solutions (three of them are one parameter solutions) to the above equations, which the authors of [19] denoted by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and \mathcal{C}_4 (\mathcal{C}_4 is valid only for $a^2 = 0$). In this paper we will be interested in the original κ -Minkowski algebra, that is $a^2 \neq 0$, and we are left with just three special cases of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. So, if we take $a^2 \neq 0$ we get three families of algebras which we denote by² $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D}_3 :

$$\mathcal{D}_1 : \quad [\hat{\xi}^{\mu}, \hat{x}^{\nu}] = i\frac{s}{a^2}a^{\mu}a^{\nu}(a\hat{\xi}) - ia^{\nu}\hat{\xi}^{\mu} \quad (11)$$

$$\mathcal{D}_2 : \quad [\hat{\xi}^{\mu}, \hat{x}^{\nu}] = i\frac{s}{a^2}a^{\mu}a^{\nu}(a\hat{\xi}) - isa^{\nu}\hat{\xi}^{\mu} + i(1-s)a^{\mu}\hat{\xi}^{\nu} \quad (12)$$

$$\mathcal{D}_3 : \quad [\hat{\xi}^{\mu}, \hat{x}^{\nu}] = i\frac{s}{a^2}a^{\mu}a^{\nu}(a\hat{\xi}) - i(1+s)\eta^{\mu\nu}(a\hat{\xi}) - ia^{\nu}\hat{\xi}^{\mu} \quad (13)$$

For $a^{\mu} = (\frac{1}{\kappa}, \vec{0})$ algebras \mathcal{D}_1 and \mathcal{D}_2 can be found in [22]. For $s = 1$ we see that algebras $\mathcal{D}_1^{s=1}$ and $\mathcal{D}_2^{s=1}$ coincide. This case was in detail investigated in [23]. In [24] (see Corollary 5.1.) the cases $\mathcal{D}_1^{s=0}$ and $\mathcal{D}_2^{s=0}$ were obtained from a different construction.

In this paper we will focus on the differential algebra \mathcal{D}_1 , since the algebras $\mathcal{D}_{2,3}$ will only lead to degenerate central metric. For $a^{\mu} = (\frac{1}{\kappa}, \vec{0})$, \mathcal{D}_1 family of algebra of differential forms has the following commutation relations with κ -Minkowski coordinates (3):

$$\left[\hat{\xi}^0, \hat{x}^0\right] = \frac{i}{\kappa}s\hat{\xi}^0; \quad \left[\hat{\xi}^k, \hat{x}^0\right] = -\frac{i}{\kappa}\hat{\xi}^k; \quad \left[\hat{\xi}^{\mu}, \hat{x}^j\right] = 0 \quad (14)$$

where $s \in \mathbb{R}$ is a free parameter.

¹ for more details see [19].

² Where \mathcal{D} stands for *differential algebra* and was already used in [22].

Let us consider a one-parameter family of Drinfeld Jordanian twist (cf. [29])

$$\mathcal{F} = \exp \left\{ \ln Z \otimes \left(\frac{1}{s-1} L_i^i - L_0^0 \right) \right\} \quad (15)$$

where $Z = [1 + \frac{s-1}{\kappa} P_0]$ and L_β^α are generators of the Lie algebra \mathfrak{igl} of inhomogeneous general linear transformations with the commutation relations $[L_\nu^\mu, L_\lambda^\rho] = \delta_\nu^\rho L_\lambda^\mu - \delta_\lambda^\mu L_\nu^\rho$; $[L_\nu^\mu, P_\lambda] = -\delta_\lambda^\mu P_\nu$. These twists provide from one hand κ -Minkowski spacetime algebra (3) and from the other a family of differential calculus (14) in such a way that they are bicovariant with respect to the action of $U^\mathcal{F}(\mathfrak{igl})$ -Hopf algebra (for more details see [19], [23]).

III. CENTRAL CO-TETRAD AND THE CORRESPONDING DEFORMED METRIC

In this section we want to build up the quantum metric tensor from quantum (noncommutative) co-tetrad $\hat{\omega}^a$:

$$\hat{g} = \eta_{ab} \hat{\omega}^a \otimes \hat{\omega}^b = \hat{\omega}^a \otimes \hat{\omega}_a \quad (16)$$

where the flat classical metric $\eta_{ab} = \text{diag}(-, +, +, +)$ is assumed to bear Lorentzian signature. In the classical limit co-tetrad consists of four linearly independent one-forms $\omega^a = e_\mu^a dx^\mu$ which by construction are orthonormal with respect to the metric (16). The dual object composed of vector fields (named tetrad or vierbein): $e_a = e_\mu^a \partial_\mu$ stands for famous E. Cartan *repère mobile*. The matrices defining the tetrad e_μ^a and co-tetrad e_a^μ have to be mutually inverse each other. In fact, this approach provides an effective link between flat and curved spacetime formalism and is also useful in noncommutative setting (see e.g. [30] and references therein). In addition the metric signature is controlled by the signature of a flat metric.

In [25] the authors investigated the differential calculi compatible with κ -Minkowski algebra of the classical dimension. In our notation this type of differential calculus corresponds to the family $\mathcal{D}_1^{s=0}$. Later on [26] they also investigated certain 2-dimensional differential calculi and they extended their investigations to \mathcal{D}_1 and \mathcal{D}_2 families (which was called α and β family there). They showed that the \mathcal{D}_2 case leads to the degenerate central metrics, except for the case: $\mathcal{D}_1^{s=1} \equiv \mathcal{D}_2^{s=1}$. In our approach also the family \mathcal{D}_3 lead to degenerate metric as well. Therefore in this section we look for other solutions in the \mathcal{D}_1 family.

We define the central co-tetrad $\hat{\omega}^a$ as a collection of four linearly-independent one-forms that commute with all the noncommutative coordinates \hat{x}^μ i.e.

$$[\hat{\omega}^a, \hat{x}^\mu] = 0. \quad (17)$$

and $\hat{\omega}^a$ can be written as

$$\hat{\omega}^a = e_\mu^a \hat{\xi}^\mu, \quad (18)$$

where the components e_μ^a are functions of noncommutative coordinates ($\in \hat{\mathcal{A}}$) that are yet to be determined. Moreover, in the classical limit the matrix e_μ^a has to be invertible (which provides an additional condition on functions e_μ^a).

In the algebra of κ -Minkowski coordinates the commutator (17) is given by:

$$[\hat{\omega}(\hat{x}^\mu), \hat{x}^0] = -\frac{i}{\kappa} \sum_i \hat{x}^i \frac{\partial}{\partial \hat{x}^i} \hat{\omega}(\hat{x}^j, \hat{x}^0); \quad [\hat{\omega}(\hat{x}^\mu), \hat{x}^k] = \hat{x}^k \left(\omega \left(\hat{x}^i, \hat{x}_0 + \frac{i}{\kappa} \right) - g(\hat{x}^\mu) \right) \quad (19)$$

We start with a single central one-form $\hat{\omega} = e_\mu \hat{\xi}^\mu$ such that $[\hat{\omega}, \hat{x}^\alpha] = 0$.

$$[\hat{\omega}, \hat{x}^0] = [e_\mu \hat{\xi}^\mu, \hat{x}^0] = e_0 [\hat{\xi}^0, \hat{x}^0] + e_k [\hat{\xi}^k, \hat{x}^0] + [e_0, \hat{x}^0] \hat{\xi}^0 + [e_k, \hat{x}^0] \hat{\xi}^k = 0 \quad (20)$$

$$[\hat{\omega}, \hat{x}^j] = [e_\mu \hat{\xi}^\mu, \hat{x}^j] = e_0 [\hat{\xi}^0, \hat{x}^j] + e_k [\hat{\xi}^k, \hat{x}^j] + [e_0, \hat{x}^j] \hat{\xi}^0 + [e_k, \hat{x}^j] \hat{\xi}^k = 0 \quad (21)$$

After using the relations (14) of differential calculus we get the following conditions:

$$[e_0, \hat{x}^0] = -\frac{i}{\kappa} s e_0 \quad ; \quad [e_k, \hat{x}^0] = \frac{i}{\kappa} e_k \quad (22)$$

and

$$[e_0, \hat{x}^j] = 0 \quad ; \quad [e_k, \hat{x}^j] = 0 \quad (23)$$

Therefore for the algebra \mathcal{D}_1 the requirement that the tetrad $\hat{\omega}$ is a central element in $\hat{\mathcal{A}}$ leads to the following commutation relations

$$\begin{aligned} [e_0, \hat{x}^0] &= -\frac{i}{\kappa} s e_0 \quad ; \quad [e_0, \hat{x}^k] = 0 \\ [e_k, \hat{x}^0] &= \frac{i}{\kappa} e_k \quad ; \quad [e_k, \hat{x}^i] = 0 \end{aligned} \quad (24)$$

It turns out that all e_μ are only functions of \hat{x}^i (as they commute with \hat{x}^j) and have to satisfy the following differential equations (cf. 19):

$$\hat{x}^k \frac{\partial}{\partial \hat{x}^k} e_0 = s e_0 \quad (25)$$

$$\hat{x}^k \frac{\partial}{\partial \hat{x}^k} e_j = -e_j \quad (26)$$

The solutions of (25) and (26) are (see also Appendix 1):

$$e_0 = \hat{r}^s E_0 \left(\frac{\hat{x}^k}{\hat{r}} \right) \quad (27)$$

$$e_j = \hat{r}^{-1} E_j \left(\frac{\hat{x}^k}{\hat{r}} \right) \quad (28)$$

So the central one-form reads as:

$$\hat{\omega} = \hat{r}^s E_0 \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^0 + \hat{r}^{-1} E_j \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^j \quad (29)$$

Similarly for the collection of four linearly independent one-forms satisfying $[\hat{\omega}^a, \hat{x}^\alpha] = 0$ we get the following solution:

$$\hat{\omega}^a = \hat{r}^s E_0^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^0 + \hat{r}^{-1} E_j^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^j \quad (30)$$

because

$$e_0^a = \hat{r}^s E_0^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \quad ; \quad e_j^a = \hat{r}^{-1} E_j^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \quad (31)$$

In result we have 16 arbitrary functions $E_\mu^a \left(\frac{\hat{x}^k}{\hat{r}} \right)$ of variables $\frac{\hat{x}^k}{\hat{r}}$ and $\hat{r} = \sqrt{\hat{x}^k \hat{x}^k}$ (note that in spherical coordinates such functions will only depend on the angles).

Now the metric can be built up from the central tetrads (16) as follows:

$$\hat{g} = \left(\hat{r}^s E_0^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^0 + \hat{r}^{-1} E_i^a \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^i \right) \otimes \left(\hat{r}^s E_{a0} \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^0 + \hat{r}^{-1} E_{aj} \left(\frac{\hat{x}^k}{\hat{r}} \right) \hat{\xi}^j \right) = \quad (32)$$

$$= \hat{r}^{2s} E_0^a E_{a0} \hat{\xi}^0 \otimes \hat{\xi}^0 + \hat{r}^{s-1} E_{a0} E_j^a \hat{\xi}^0 \otimes \hat{\xi}^j + \hat{r}^{s-1} E_i^a E_{a0} \hat{\xi}^i \otimes \hat{\xi}^0 + \hat{r}^{-2} E_i^a E_{aj} \hat{\xi}^i \otimes \hat{\xi}^j \quad (33)$$

One can show that this type of the metric belongs to the center of the algebra $\hat{\mathcal{A}}$ of κ -Minkowski type (3) as well and has vanishing commutators:

$$[\hat{g}, \hat{x}^\mu] = 0 \quad (34)$$

i.e. it falls in the framework introduced in [25].

A. Classical limit

In the classical limit $\kappa \rightarrow \infty$ when the noncommutative objects (coordinates, differentials etc.) become commutative, the functions E_μ^a will become the arbitrary functions of $\frac{x^k}{r}$ and $r = \sqrt{x^k x^k}$ commutative coordinates.

In this case the metric for algebra \mathcal{D}_1 (14) in the classical limit reads as:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \frac{1}{r^2} E_i^a E_{aj} dx^i \otimes dx^j + r^{(s-1)} E_i^a E_{a0} (dx^i \otimes dt + dt \otimes dx^i) + r^{2s} E_0^a E_{a0} dt \otimes dt \quad (35)$$

One can see that in the above metric the functions E_μ^a do not depend on time therefore such metrics could describe only stationary solutions.

When we introduce the spherical coordinates

$$(t, r, \theta, \phi) : \quad x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

the functions $E_\mu^a(\theta, \phi)$ are arbitrary functions of the angles (θ, ϕ) only. Therefore, in what follows, we shall use spherical coordinate system to write down the metric in the form: ³

$$g = \begin{pmatrix} r^{2s} E_0^a(\theta, \phi) E_{a0}(\theta, \phi) & r^{(s-1)} E_1^a(\theta, \phi) E_{a0}(\theta, \phi) & r^s E_2^a(\theta, \phi) E_{a0}(\theta, \phi) & r^s E_3^a(\theta, \phi) E_{a0}(\theta, \phi) \\ r^{(s-1)} E_1^a(\theta, \phi) E_{a0}(\theta, \phi) & \frac{E_1^a(\theta, \phi) E_{a1}(\theta, \phi)}{r^2} & \frac{E_1^a(\theta, \phi) E_{a2}(\theta, \phi)}{r} & \frac{E_1^a(\theta, \phi) E_{a3}(\theta, \phi)}{r} \\ r^s E_2^a(\theta, \phi) E_{a0}(\theta, \phi) & \frac{E_1^a(\theta, \phi) E_{a2}(\theta, \phi)}{r} & E_2^a(\theta, \phi) E_{a2}(\theta, \phi) & E_2^a(\theta, \phi) E_{a3}(\theta, \phi) \\ r^s E_3^a(\theta, \phi) E_{a0}(\theta, \phi) & \frac{E_1^a(\theta, \phi) E_{a3}(\theta, \phi)}{r} & E_2^a(\theta, \phi) E_{a3}(\theta, \phi) & E_3^a(\theta, \phi) E_{a3}(\theta, \phi) \end{pmatrix}$$

We are looking for non degenerate solutions so from the condition on non-zero determinant $\det(g) \neq 0$ we get additional constraints on the functions E_μ^a .

We can simplify the notation by introducing

$$r^{-2} E_i^a(\theta, \phi) E_{aj}(\theta, \phi) = r^{-2} a_{ij}(\theta, \phi) \quad (36)$$

$$r^{s-1} E_0^a(\theta, \phi) E_{aj}(\theta, \phi) = r^{s-1} b_j(\theta, \phi) \quad (37)$$

$$r^{2s} E_0^a(\theta, \phi) E_{a0}(\theta, \phi) = r^{2s} c(\theta, \phi) \quad (38)$$

The Einstein equations are written in the form (G being the universal Newtonian constant of gravity):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \quad (39)$$

Given a specified distribution of matter and energy $T_{\mu\nu}$, the equations (39) are understood to be equations for the metric tensor $g_{\mu\nu}$, as both the Ricci tensor $R_{\mu\nu}$ and scalar curvature R depend on the metric in a complicated nonlinear manner. One can write the Einstein equations in a more compact form by introducing the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. Since in the classical limit our metric has to satisfy the Einstein equations, this leads us to some coupled differential equations for tetrad functions E_μ^a which turn out to be non trivial to solve. In order to find some solutions, we consider a special case with the 'constant coefficients', i.e. such that the metric depends only on r coordinate and:

$$E_i^a E_{aj} = a_{ij} = \text{const} \quad ; \quad E_{a0} E_j^a = b_j = \text{const} \quad ; \quad E_0^a E_{a0} = c = \text{const} \quad (40)$$

The metric looks as follows:

$$g = \begin{pmatrix} r^{2s} c & r^{(s-1)} b_1 & r^s b_2 & r^s b_3 \\ r^{(s-1)} b_1 & \frac{a_{11}}{r^2} & \frac{a_{12}}{r} & \frac{a_{13}}{r} \\ r^s b_2 & \frac{a_{12}}{r} & a_{22} & a_{23} \\ r^s b_3 & \frac{a_{13}}{r} & a_{23} & a_{33} \end{pmatrix} \quad (41)$$

and the determinant: $r^{2s-2} \{b_1^2 a_{23}^2 - b_1^2 a_{22} a_{33} + b_2^2 a_{13}^2 - b_2^2 a_{11} a_{33} + b_3^2 a_{12}^2 - b_3^2 a_{11} a_{22} - c a_{11} a_{23}^2 - c a_{13}^2 a_{22} - c a_{12}^2 a_{33} + c a_{11} a_{22} a_{33} + 2c a_{12} a_{13} a_{23} + 2b_1 b_2 a_{12} a_{33} - 2b_1 b_2 a_{13} a_{23} - 2b_1 b_3 a_{12} a_{23} + 2b_1 b_3 a_{13} a_{22} + 2b_2 b_3 a_{11} a_{23} - 2b_2 b_3 a_{12} a_{13}\} \neq 0$ is assumed to be non-zero.

In the following we will look for the solutions of non-vacuum $G^{\mu\nu} = 8\pi G T^{\mu\nu}$ and vacuum $G^{\mu\nu} = 0$ Einstein equations.

³ In fact, the coefficient functions $E_\mu^a(\theta, \phi)$ appearing in formulas (35) and below are in general not identical. They can be express each other as linear combinations with trigonometric functions of (θ, ϕ) .

1. Non vacuum solutions

In the tetrad formalism we can work with Einstein equations in the Lorentzian frame: $G^{ab} = 8\pi G T^{ab}$. The passage to the coordinate frame is determined by standard formulae: $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ with $\eta_{ab} = \text{diag}(-, +, +, +)$ and $G_\nu^\mu = e_a^\mu e_\nu^b G_b^a$, etc.

On the left hand side of the non-vacuum Einstein equations we assume the energy-momentum of perfect and isotropic fluid, i.e. $T^{ab} = (\rho + p)u^a u^b + p\eta^{ab} = \text{diag}(\rho, p, p, p)$ where ρ is energy density, p is the pressure of the fluid and the vector u^a represents its four-velocity. Assuming further that the fluid is co-moving with respect to the tetrad, i.e. $u^a = (1, 0, 0, 0)$ we can diagonalize the energy-momentum tensor $T_b^a = \text{diag}(-\rho, p, p, p)$. Therefore to look for solutions with perfect fluid one should firstly diagonalize the Einstein tensor G_μ^ν .⁴ For this purpose we calculate the characteristic polynomial: $\det[G_\mu^\nu - \lambda\delta_\mu^\nu] \Leftrightarrow \det[G_{\mu\nu} - \lambda g_{\mu\nu}] = 0$ which will give us the diagonal form of the G_b^a with the roots of this equation on the diagonal. Having the multiplicity 1 of one solution $\tilde{\lambda}$ and multiplicity 3 of another λ will allow us to write down the equation with the (diagonal) momentum energy tensor for the perfect fluid as:

$$G_b^a = \begin{pmatrix} \tilde{\lambda} & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix} = 8\pi G \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \quad (42)$$

The equation of state of barotropic fluid is characterized by a dimensionless number w equal to the ratio of its pressure p to the energy density ρ : $w = p/\rho = -\lambda/\tilde{\lambda}$. For example the most known from cosmology cases are:

- cosmological constant (dark energy) which corresponds to $w = -1$,
- dust matter (dark or/and ordinary baryonic matter) ($w = 0$),
- radiation ($w = 1/3$).

It turns out that spatial curvature of FLRW metric can be also described by the barotropic factor $w = -\frac{1}{3}$ satisfying strong energy condition $\rho + 3p \geq 0$.

Therefore after the diagonalization of the Einstein tensor we can see which kind of the equation of state can be derived from the quantum metric (41). In the case under consideration one obtains only two possible solution of the barotropic type (there is no other solutions).

1. Quantum Universe with spatial curvature type barotropic factor $w = -1/3$;

$$(G_b^a)_I = \begin{pmatrix} \lambda^I & 0 & 0 & 0 \\ 0 & \frac{1}{3}\lambda^I & 0 & 0 \\ 0 & 0 & \frac{1}{3}\lambda^I & 0 \\ 0 & 0 & 0 & \frac{1}{3}\lambda^I \end{pmatrix} = 8\pi G \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$(G^{ab})_I = \begin{pmatrix} -\lambda^I & 0 & 0 & 0 \\ 0 & \frac{1}{3}\lambda^I & 0 & 0 \\ 0 & 0 & \frac{1}{3}\lambda^I & 0 \\ 0 & 0 & 0 & \frac{1}{3}\lambda^I \end{pmatrix}$$

where $\lambda^I = -\frac{a_{22}s^2}{4(a_{12}^2 - a_{11}a_{22})}$. This eigenvalue is given for simplified metric with the choice $c = 0; b_1 = 0; b_2 = 0$,

which does not change the barotropic factor. The full expression for λ^I corresponding exactly to (41) can be found in Appendix 2. From "polynomial constraints" (see Appendix 2) it follows that $a_{23} = \sqrt{a_{22}a_{33}}$. The determinant has to be non-zero $\det(g_{I_{simpl}}) = r^{2s-2}b_3^2(a_{12}^2 - a_{11}a_{22}) \neq 0$ and $\det(g_{I_{simpl}}) < 0$ iff $a_{12}^2 < a_{11}a_{22}$. The corresponding metric is then:

$$g_{I_{simpl}} = \begin{pmatrix} 0 & 0 & 0 & r^s b_3 \\ 0 & \frac{a_{11}}{r^2} & \frac{a_{12}}{r} & \frac{a_{13}}{r} \\ 0 & \frac{a_{12}}{r} & a_{22} & \sqrt{a_{22}a_{33}} \\ r^s b_3 & \frac{a_{13}}{r} & \sqrt{a_{22}a_{33}} & a_{33} \end{pmatrix} \quad (43)$$

⁴ The Lorentz indices a, b, \dots are raised and lowered by means of the flat metric and its inverse.

2. Universe with dark energy (cosmological constant) with $w = -1$;

$$(G_b^a)_{II} = \begin{pmatrix} \lambda^{II} & 0 & 0 & 0 \\ 0 & \lambda^{II} & 0 & 0 \\ 0 & 0 & \lambda^{II} & 0 \\ 0 & 0 & 0 & \lambda^{II} \end{pmatrix} = 8\pi G \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$(G^{ab})_{II} = \begin{pmatrix} -\lambda^{II} & 0 & 0 & 0 \\ 0 & \lambda^{II} & 0 & 0 \\ 0 & 0 & \lambda^{II} & 0 \\ 0 & 0 & 0 & \lambda^{II} \end{pmatrix}$$

where $\lambda^{II} = \frac{s^2}{2a_{11}}$ (is the simplified version for the choice of the coefficients $a_{12} = 0; a_{23} = 0; a_{13} = 0; b_1 = 0 = b_3$). Note that $s \neq 0$. The expression for λ_2 depending on all constant coefficients is given in the Appendix 2 as well. The metric itself looks as follows:

$$g_{II_{simpl}} = \begin{pmatrix} r^{2s} \frac{b_2^2}{2a_{22}} & 0 & r^s b_2 & 0 \\ 0 & \frac{a_{11}}{r^2} & 0 & 0 \\ r^s b_2 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix} \quad (44)$$

The determinant of the metric: $\det(g_{II_{simpl}}) = -\frac{1}{2}b_2^2 a_{33} a_{11} \neq 0$ and $\det g_{II_{simpl}}$ can be chosen to be negative for $a_{33} a_{11} > 0$.

However, choosing $s = 0$ in this case, i.e. $\begin{pmatrix} \frac{b_2^2}{2a_{22}} & 0 & b_2 & 0 \\ 0 & \frac{a_{11}}{r^2} & 0 & 0 \\ b_2 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$ we get an example of vacuum (but flat) solution

of Einstein equations, for which $G_{\mu\nu} = 0$ (note that $G^{\mu\nu} = 0 \Rightarrow R^{\mu\nu} = 0$). Below we present yet other interesting and not flat vacuum solutions, both for $s \neq 0$ and $s = 0$ cases.

2. Vacuum solutions

1. Vacuum solution of the pp-wave type.

One of the examples of the vacuum solution can be provided for the special choice of non-zero, constant coefficients b_1, a_{22}, a_{33} and restoring the dependence on the angles in the coefficient $a_{11}(\phi, \theta)$ i.e. choosing the metric as:

$$g_{1vac} = \begin{pmatrix} 0 & r^{(s-1)}b_1 & 0 & 0 \\ r^{(s-1)}b_1 & \frac{a_{11}(\phi, \theta)}{r^2} & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix} \quad (45)$$

Then the corresponding Einstein tensor vanishes for any choice of the parameter $s \neq 0$ but under the following additional conditions on the derivatives of function $a_{11}(\phi, \theta)$ as follows: $\frac{\partial a_{11}}{\partial \phi} \neq 0; \frac{\partial a_{11}}{\partial \theta} \neq 0; \frac{\partial^2 a_{11}}{\partial \phi \partial \theta} \neq 0$ and $\frac{\partial^2 a_{11}}{\partial \phi^2} = 0 = \frac{\partial^2 a_{11}}{\partial \theta^2}$ [which amounts to $a_{11}(\phi, \theta) = \xi_1 + \xi_2 \theta + \xi_3 \phi + \xi_4 \theta \phi$, ξ_i are constants]. The Riemann tensor however does not vanish: $R_{t\theta\phi r} = \frac{r^{-(s+1)}}{2b_1} \frac{\partial^2 a_{11}}{\partial \phi \partial \theta} = R_{t\phi\theta r}; R_{\theta r\phi r} = -\frac{1}{2a_{22}r^2} \frac{\partial^2 a_{11}}{\partial \phi \partial \theta}; R_{\phi r\theta r} = -\frac{1}{2a_{33}r^2} \frac{\partial^2 a_{11}}{\partial \phi \partial \theta}$. Therefore (45) constitutes vacuum but non-flat solution.

2. Another example of vacuum solution, also of pp-wave type can be provided (for $s = 0$), by:

$$g_{2vac} = \begin{pmatrix} c(\phi, \theta) & r^{(-1)}b_1 & 0 & 0 \\ r^{(-1)}b_1 & 0 & 0 & 0 \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix} \quad (46)$$

The corresponding Einstein tensor vanishes by imposing additional conditions (analogous to the above ones) on the derivatives of function $c(\phi, \theta)$ as follows: $\frac{\partial c}{\partial \phi} \neq 0; \frac{\partial c}{\partial \theta} \neq 0; \frac{\partial^2 c}{\partial \phi \partial \theta} \neq 0$ and $\frac{\partial^2 c}{\partial \phi^2} = 0 = \frac{\partial^2 c}{\partial \theta^2}$ [which amounts

to $c(\phi, \theta) = \zeta_1 + \zeta_2\theta + \zeta_3\phi + \zeta_4\theta\phi$, ζ_i are constants]. However the Riemann tensor does not vanish ($R_{r\theta\phi t} = \frac{r}{2b_1}\frac{\partial^2 c}{\partial\phi\partial\theta} = R_{r\phi\theta t}$; $R_{\theta t\phi t} = -\frac{1}{2a_{22}}\frac{\partial^2 c}{\partial\phi\partial\theta}$; $R_{\phi t\theta t} = -\frac{1}{2a_{33}}\frac{\partial^2 c}{\partial\phi\partial\theta}$). Therefore (46) is as well vacuum non-flat (Ricci-flat) solution.

It is rather known that the so called pp-waves (plane-fronted waves with parallel rays) are exact wave like solutions of Einstein equations which represent gravitational radiation propagating with the speed of light in a direction determined by light-like Killing vector field. They are analogous to source-free photons in Maxwell electrodynamics. They also appear in many places of Theoretical Physics e.g. as string or D-brane background, and supersymmetry, etc..

Note that the non-vacuum and vacuum solutions for $s = 0$ are interesting from the quantum symmetry point of view, since in this case the Jordanian twist (15) reduces to

$$\mathcal{F} = \exp \left\{ \ln \left(1 - \frac{1}{\kappa} P_0 \right) \otimes D \right\} \quad (47)$$

and corresponds to Poincaré-Weyl symmetry (one generator extension of the Poincaré algebra, with adding dilatation generator D) as a minimal quantum group providing symmetry algebra [29].

IV. FINAL REMARKS

We have investigated some of the properties of quantum spaces using the central co-tetrad formalism. We used a sort of a toy model for quantum gravity effects, which should be encoded in the noncommutative κ -Minkowski algebra (3) and by analyzing a certain bicovariant differential calculus (compatible with such noncommutative structure) we found equations for the components of the noncommutative central co-tetrads, which we solve in general (30). We show that this formalism gives rise to the same quantum central metric as proposed in [25], but here the benefit is that the Lorentzian signature is built in from the very beginning.

Analyzing the classical limit of our quantum metric, and by imposing the validity of Einstein equations in that limit, we found (under further assumptions which simplify the calculations) new vacuum and non-vacuum solutions. Namely, the solutions of Einstein equations for our simplified cases (metric with constant coefficients and only r -dependence) contribute to the description of the (quantum) Universe with the cosmological constant and the spacial curvature. Also, a vacuum solution corresponding to pp-wave spacetime is obtained.

The idea is to investigate more complicated and more general solution of (30). One can see that in the quantum metric (30), the functions E_μ^a do not depend on time therefore such metrics could describe only static solutions for \mathcal{D}_1 . Maybe for more general differential calculi $\mathcal{C}_1, \dots, \mathcal{C}_4$ (classified in [19]) one can recover some new interesting cosmological and maybe even black hole solutions.

Appendix 1

We want to find the general solution for the following differential equation

$$\hat{x}^i \frac{\partial}{\partial \hat{x}^i} f = \gamma f \quad (48)$$

We denote the dilatation operator $\hat{D} = \hat{x}^i \frac{\partial}{\partial \hat{x}^i}$. For any dimension we can define the spherical coordinates and write the radial vector and nabla operators as

$$\vec{\hat{r}} = \hat{x}_k \vec{e}_k = \hat{r} \vec{\hat{r}}_0 \quad \vec{\nabla} = \vec{e}_k \frac{\partial}{\partial \hat{x}_k} = \vec{\hat{r}}_0 \frac{\partial}{\partial \hat{r}} + \text{terms in directions perpendicular to } \vec{\hat{r}}_0 \quad (49)$$

where \vec{e}_k are constant orthonormal vectors of basis in Cartesian coordinate system and $\vec{\hat{r}}_0$ is radial unit vector. Now, for the dilatation operator we have

$$\hat{D} = \hat{x}^i \frac{\partial}{\partial \hat{x}^i} = \vec{\hat{r}} \cdot \vec{\nabla} = \hat{r} \frac{\partial}{\partial \hat{r}} \quad (50)$$

so, differential equation (48) can be solved by direct integration which gives the following solution

$$f = \text{const. } \hat{r}^\gamma \quad (51)$$

Notice that $\hat{D}\frac{x_k}{\hat{r}} = 0$, and so any arbitrary function of $\frac{x_k}{\hat{r}}$ satisfies $\hat{D}F(\frac{x_k}{\hat{r}}) = 0$ the most general “const.” is $F(\frac{x_k}{\hat{r}})$ which gives the most general solution for (48) is

$$f = \hat{r}^\gamma F\left(\frac{x_k}{\hat{r}}\right) \quad (52)$$

This enables us to solve differential equations.

Appendix 2: Full solution for the metric (41)

In section III.A.1 we focused on non vacuum solutions with isotropic metric (41) , i.e. for $a_{ij} = \text{const}, b_i = \text{const}$ and $c = \text{const}$) and only with dependence on r :

$$g = \begin{pmatrix} r^{2s}c & r^{(s-1)}b_1 & r^s b_2 & r^s b_3 \\ r^{(s-1)}b_1 & \frac{a_{11}}{r^2} & \frac{a_{12}}{r} & \frac{a_{13}}{r} \\ r^s b_2 & \frac{a_{12}}{r} & a_{22} & a_{23} \\ r^s b_3 & \frac{a_{13}}{r} & a_{23} & a_{33} \end{pmatrix} \quad (53)$$

After diagonalization of the Einstein tensor G_ν^μ wrt this metric we obtain only two possible solutions of the perfect fluid type (there is no other solutions). The characteristic equation is of the form : $\det[G_\nu^\mu - \lambda\delta_\nu^\mu] = \alpha \cdot \beta \cdot \gamma = 0$ where:

$$\begin{aligned} \alpha &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2) s^2 + x\lambda_1)^2 \\ \beta &= ((-3a_{33}b_2^2 + 6a_{23}b_2b_3 - 3a_{22}b_3^2 - 4a_{23}^2c + 4a_{22}a_{33}c) s^2 + x\lambda_2) \\ \gamma &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2 - 4a_{23}^2c + 4a_{22}a_{33}c) s^2 + x\lambda_3) \end{aligned}$$

$$\begin{aligned} \text{and } x &= -4(a_{23}^2 - a_{22}a_{33})b_1^2 - 4(a_{12}^2 - a_{11}a_{22})b_3^2 - 4(a_{13}^2 - a_{11}a_{33})b_2^2 \\ &+ 8a_{13}a_{23}b_1b_2 - 8a_{12}a_{33}b_1b_2 - 8a_{13}a_{22}b_1b_3 + 8a_{12}a_{23}b_1b_3 \\ &+ 8a_{12}a_{13}b_2b_3 - 8a_{11}a_{23}b_2b_3 + 4(a_{13}^2a_{22} - 2a_{12}a_{13}a_{23} + a_{11}a_{23}^2 + a_{12}^2a_{33} - a_{11}a_{22}a_{33})c \end{aligned}$$

I. To find the solution of multiplicity 3 and 1 (corresponding to subcase 1. in Sec.III.A.1) we notice that for: $-4a_{23}^2c + 4a_{22}a_{33}c = 0$ we have the following:

$$\begin{aligned} \alpha &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2) s^2 + x\lambda_1^I)^2 \\ \beta &= ((-3a_{33}b_2^2 + 6a_{23}b_2b_3 - 3a_{22}b_3^2) s^2 + x\lambda_2^I) \\ \gamma &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2) s^2 + x\lambda_3^I) \end{aligned}$$

Therefore $\lambda_1^I = \lambda_3^I$ (the root of multiplicity 3) and the characteristic equation is as follows:

$$(((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2) s^2 + x\lambda_1^I)^3 (3(-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2) s^2 + x\lambda_2^I) = 0.$$

It leads to: $\lambda_1^I = \frac{(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)s^2}{x}$ as the triple multiplicity root and $\lambda_2^I = \frac{3(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)s^2}{x} = 3\lambda_1^I$ as a single multiplicity root for $s \neq 0$.

The additional condition $-4a_{23}^2c + 4a_{22}a_{33}c = 0$ we shall call "polynomial constraint" which relates the coefficients of the metric in the following way:

$$a_{23}^2 = a_{22}a_{33} \quad (54)$$

This results in $\lambda_2^I = 3\lambda_1^I$ and therefore barotropic factor $w = -1/3$ (Quantum Universe with spatial curvature type barotropic factor):

$$(G^{ab})_I = \begin{pmatrix} -\lambda^I & 0 & 0 & 0 \\ 0 & \frac{1}{3}\lambda^I & 0 & 0 \\ 0 & 0 & \frac{1}{3}\lambda^I & 0 \\ 0 & 0 & 0 & \frac{1}{3}\lambda^I \end{pmatrix} \quad (55)$$

And the corresponding metric tensor:

$$g_I = \begin{pmatrix} r^{2s}c & r^{(s-1)}b_1 & r^s b_2 & r^s b_3 \\ r^{(s-1)}b_1 & \frac{a_{11}}{r^2} & \frac{a_{12}}{r} & \frac{a_{13}}{r} \\ r^s b_2 & \frac{a_{12}}{r} & a_{22} & \sqrt{a_{22}a_{33}} \\ r^s b_3 & \frac{a_{13}}{r} & \sqrt{a_{22}a_{33}} & a_{33} \end{pmatrix} \quad (56)$$

II. Another solution is obtained with the polynomial constraint of the form: $-4a_{23}^2c + 4a_{22}a_{33}c = 2(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)$

With this we get the following:

$$\begin{aligned} \alpha &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2)s^2 + x\lambda_1)^2 \\ \beta &= ((-a_{33}b_2^2 + 2a_{23}b_2b_3 - a_{22}b_3^2)s^2 + x\lambda_2) \\ \gamma &= ((a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)s^2 + x\lambda_3) \end{aligned}$$

which leads to:

- the root of multiplicity 3 is $\lambda_1^{II} = \frac{(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)s^2}{x} = \lambda_2^{II}$

- the root of multiplicity 1 is $\lambda_3^{II} = -\frac{(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)s^2}{x} = -\lambda_1^{II}$

and contributes to a barotropic factor $w = -1$ (Universe with dark energy - cosmological constant) with the following Einstein tensor:

$$(G^{ab})_{II} = \begin{pmatrix} -\lambda^{II} & 0 & 0 & 0 \\ 0 & \lambda^{II} & 0 & 0 \\ 0 & 0 & \lambda^{II} & 0 \\ 0 & 0 & 0 & \lambda^{II} \end{pmatrix} \quad (57)$$

From the additional condition $-4a_{23}^2c + 4a_{22}a_{33}c = 2(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)$ we get the relation for the coefficients of the metric in the following way:

$$c = -\frac{(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)}{2(a_{23}^2 - a_{22}a_{33})} \quad (58)$$

Therefore the metric corresponding to this case is:

$$g_{II} = \begin{pmatrix} -r^{2s} \frac{(a_{33}b_2^2 - 2a_{23}b_2b_3 + a_{22}b_3^2)}{2(a_{23}^2 - a_{22}a_{33})} & r^{(s-1)}b_1 & r^s b_2 & r^s b_3 \\ r^{(s-1)}b_1 & \frac{a_{11}}{r^2} & \frac{a_{12}}{r} & \frac{a_{13}}{r} \\ r^s b_2 & \frac{a_{12}}{r} & a_{22} & a_{23} \\ r^s b_3 & \frac{a_{13}}{r} & a_{23} & a_{33} \end{pmatrix} \quad (59)$$

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